Multi-Asset Options.

We have seen already a European Call Option on an asset S with strike K, expiry at time T and payoff,

$$(S_T - K)^+$$

This amounts to an option to exchange K for S at time T, so far as the holder of the option is concerned. One can have an option to exchange an asset S^2 for an asset S^1 at time T. This will have payoff $(S^1_T - S^2_T)^+$ and clearly it will be exercised only on the set $\{S^1_T \geq S^2_T\}$. In this situation both S^1 and S^2 are understoool to be log-normal assets, that is, for $i=1,2$,

$$S^i_t = S^i_0 + \int_0^t \mu^i S^i_s ds + \int_0^t \sigma^i S^i_s dW^i_s$$

and we assume that the Brownian Motions W^1, W^2 have constant correlation, $\rho \in [-1,+1]$. We know that this means,

$$\langle W^1, W^2 \rangle_t = \rho t .$$

Now, initially, we have a

probability measure, $\mathbb{P}$ say, and the drift and volatility of S^i is given by μ^i, σ^i under this measure $\mathbb{P}$.

We let r be the interest rate. This means that we can write

$$\tilde{S}^i_t = \tilde{S}^i_0 + \int_0^t \sigma^i \tilde{S}^i_s \, d\left(W^i_s + \frac{(\mu^i - r)}{\sigma^i} s\right)$$

and for convenience $\lambda^i = \dfrac{\mu^i - r}{\sigma^i}$.

If we can find a measure $\mathbb{Q}$ such that each of $(W^1_s + \lambda^1 s)$ and $(W^2_s + \lambda^2 s)$ are $\mathbb{Q}$ Brownian Motions then our discounted assets will be $\mathbb{Q}$-martingales and it will follow that the time $t=0$ value of the payoff $(S^1_T - S^2_T)^+$ will be given by,

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S^1_T - S^2_T)^+}{e^{rT}}\right).$$

Note also that if $\mathbb{Q}$ is such that

$$\tilde{S}^i_t = \tilde{S}^i_0 + \int_0^t \sigma^i \tilde{S}^i_s \, d\hat{W}^i_s,$$

where $\hat{W}^i$ is a $\mathbb{Q}$ Brownian Motion, then

$$S^i_t = S^i_0 + \int_0^t r S^i_s \, ds + \int_0^t \sigma^i S^i_s \, d\hat{W}^i_s$$

that is, the drift of S^i, $i=1,2$, is "r", the risk-free rate, under $\mathbb{Q}$.

But is there such a $\mathbb{Q}$? Recall that at the end of our proof of martingale representation we observed that we could extend the result to multi-dimensional processes. We saw too that there is a multi-dimensional version of Levy's Theorem. You may have guessed that there is too a multi-dimensional version of Girsanov's result. We digress!:

Let $\underline{B}_t = (B^1_t, B^2_t)$ be a two dimensional Brownian Motion adapted to some filtration satisfying the 'usual conditions'. Let us define two assets S^1, S^2, adapted to the filtration of $\underline{B}$ by, for $i=1,2$,

$$S^i_t = S^i_0 + \int_0^t \alpha^i(s) S^i_s \, ds + \int_0^t \beta_{i1}(s) S^i_s \, dB^1_s + \int_0^t \beta_{i2}(s) S^i_s \, dB^2_s$$

We are going to try to choose the coefficients α^i, β_{ij}, so that we recover the assets S^1, S^2 introd-

-uced at the start of this section. Clearly, we ~~would~~ should choose $\alpha^i(s) = \mu^i$. If we assume that $\beta_{ij}(s)$ are constants then we can rewrite the stochastic integral terms;

$$\int_0^t \beta_{i1} S^i_s dB^1_s + \int_0^t \beta_{i2} S^i dB^2_s = \int_0^t S^i_s d(\beta_{i1} B^1_s + \beta_{i2} B^2_s)$$

$$= \int_0^t S^i_s \sigma^i d\left(\frac{\beta_{i1}}{\sigma^i} B^1_s + \frac{\beta_{i2}}{\sigma^i} B^2_s\right)$$

Can we choose β_{i1}, β_{i2}, so that $\left(\frac{\beta_{i1}}{\sigma^i} B^1_s + \frac{\beta_{i2}}{\sigma^i} B^2_s\right)$ is a Brownian Motion (W^i_s) and can we choose these coefficients so that W^1 and W^2 are Brownian Motions with correlation coefficient the constant ρ? Remarkably, the answer is yes!

If we set,

$$\beta_{11} = \sigma^1 \rho \qquad \beta_{12} = \sigma^1 (1-\rho^2)^{1/2}$$

then $W^1 = \rho B^1 + (1-\rho^2)^{1/2} B^2$ and we observe that W^1 is a continuous martingale. Moreover,

$$\langle W^1, W^1 \rangle_t = \langle \rho B^1 + (1-\rho^2)^{1/2} B^2, \rho B^1 + (1-\rho^2)^{1/2} B^2 \rangle_t$$

$$= \rho^2 \langle B^1, B^1 \rangle_t + 2\rho(1-\rho^2)^{1/2} \langle B^1, B^2 \rangle_t + (1-\rho^2)\langle B^2, B^2 \rangle_t$$

$= \rho^2 t \quad + \quad 0 \quad + \quad (1-\rho^2)t \quad = t.$

So, by Levy's Theorem, W^1 is a Brownian Motion. We get W^2 by defining it to be B^1; $\beta_{21} = \sigma_1$, $\beta_{22} = 0$. So W^2 is a "BM" and,

$$\langle W^1, W^2 \rangle_t = \langle \rho B^1 + (1-\rho^2)^{\frac{1}{2}} B^2, B^1 \rangle_t$$

$$= \rho \langle B^1, B^1 \rangle_t + (1-\rho^2)^{1/2} \langle B^2, B^1 \rangle_t$$

$= \rho t$. So W^1 and W^2 have correlation ρ. To conclude, by starting with a two-dimensional Brownian motion we can construct assets S^1 and S^2 satisfying the equations introduced at the start of this section.

So we have assets,

$$S^i_t = S^i_0 + \int_0^t \mu^i S^i_s \, ds + \int_0^t \sigma^i S^i_s \, dW^i_s$$

for $i = 1, 2$, and

$$\tilde{S}^i_t = \tilde{S}^i_0 + \int_0^t \sigma^i S^i_s \, d\left(W^i_s + \left(\frac{\mu^i - r}{\sigma^i}\right)s\right).$$

Can we now find a change of measure from $\mathbb{P}$ to $\mathbb{Q}$, so that $(\tilde{S}^i_t)$ are $\mathbb{Q}$ martingales? Consider the following:

Let (U_t) be an $L^2(\mathbb{P})$ martingale and define,

$$\tilde{U}_t = U_t - \int_0^t X_s^1 d\langle U, B^1\rangle_s - \int_0^t X_s^2 d\langle U, B^2\rangle_s$$

where B^1, B^2, are as above and (X_s^1), (X_s^2) are, say, bounded continuous processes. Let

$$\mathbb{Q}(E) = \int_E e^{\int_0^T X_s^1 dB_s^1 + \int_0^T X_s^2 dB_s^2 - \frac{1}{2}\int_0^T \|\underline{X}_s\|_2^2 ds} d\mathbb{P}$$

where $\|\underline{X}_s\|_2^2 = (X_s^1)^2 + (X_s^2)^2$. We show that $\tilde{U}$ is a $\mathbb{Q}$-martingale.

Writing

$$Z(\underline{X})_t \triangleq e^{\int_0^t X_s^1 dB_s^1 + \int_0^t X_s^2 dB_s^2 - \frac{1}{2}\int_0^t \|\underline{X}_s\|_2^2 ds}$$

an application of Itô's Lemma for a semimartingales shows that

$$Z(\underline{X})_t = 1 + \int_0^t Z(\underline{X})_s X_s^1 dB_s^1 + \int_0^t Z(\underline{X})_s X_s^2 dB_s^2$$

and because $\underline{X}$ is bounded etc, $(Z(\underline{X})_t)$ is a martingale. Observe also that — using the integration by parts formula —

$$\tilde{U}_t Z(\underline{X})_t = 1\cdot U_0 + \int_0^t U_s Z(\underline{X})_s X_s^1 dB_s^1 + \int_0^t U_s Z(\underline{X})_s X_s^2 dB_s^2$$
$$+ \int_0^t Z(\underline{X})_s dU_s - \int_0^t Z(\underline{X})_s X_s^1 d\langle U, B^1\rangle_s - \int_0^t Z(\underline{X})_s X_s^2 d\langle U, B^2\rangle_s$$
$$+ \left\langle \int_0^{\cdot} Z(\underline{X})_s X_s^1 dB_s^1 + \int_0^{\cdot} Z(\underline{X})_s X_s^2 dB_s^2 + 1, U \right\rangle_t .$$

The cross variation term is equal to

$$\int_0^t Z(\underline{X})_s X_s^1 \, d\langle U, B^1\rangle_s + \int_0^t Z(X)_s X_s^2 \, d\langle U, B^2\rangle_s$$

and cancels with other terms to leave

$$\tilde{U}_t Z(\underline{X})_t = U_0 + \int_0^t U_s Z(\underline{X})_s X_s^1 \, dB_s^1 + \int_0^t U_s Z(\underline{X})_s X_s^2 \, dB_s^2$$
$$+ \int_0^t Z(\underline{X})_s \, dU_s$$

which is a $\mathbb{P}$-martingale — subject to appropriate integrability conditions which will be satisfied when we apply this result. This shows us that $(\tilde{U}_t)$ is a $\mathbb{Q}$-martingale. Applying this result to the Brownian Motions W^1, W^2 which we construct-ed earlier we see that under $\mathbb{Q}$

$$\omega_t^1 = W_t^1 - \int_0^t X_s^1 \, d\langle W^1, B^1\rangle_s - \int_0^t X_s^2 \, d\langle W^1, B^2\rangle_s$$

$$\omega_t^2 = W_t^2 - \int_0^t X_s^1 \, d\langle W^2, B^1\rangle_s - \int_0^t X_s^2 \, d\langle W^2, B^2\rangle_s$$

are $\mathbb{Q}$-martingales. Since

$$W^1 = \rho B^1 + (1-\rho^2)^{1/2} B^2$$

$$\langle W^1, B^1\rangle_t = \rho \langle B^1, B^1\rangle_t + (1-\rho^2)^{1/2} \langle B^2, B^1\rangle_t$$
$$= \rho t$$

while

$$\langle W^1, B^2\rangle_t = \rho \langle B^1, B^2\rangle_t + (1-\rho^2)^{1/2} \langle B^2, B^2\rangle_t$$

so $\langle W^1, B^2\rangle_t = (1-\rho^2)^{1/2} t$. Now $W_2 = B^1$, so,

$$\langle W^2, B^1\rangle = t \quad , \quad \langle W^2, B^2\rangle = 0$$

So, rewriting,

$$\omega^1_t = W^1_t - \int_0^t X^1_s \rho\, ds - \int_0^t X^2_s (1-\rho^2)^{1/2} ds$$

$$\omega^2_t = W^2_t - \int_0^t X^1_s\, ds$$

Choose $X^1_s \equiv \dfrac{r-\mu^2}{\sigma^2}$ then

$$\omega^2_t = W^2_t + \frac{(\mu^2 - r)}{\sigma^2} t \quad \text{and}$$

choose $X^2_s \equiv \dfrac{\dfrac{\mu^1 - r}{\sigma^1} + \dfrac{(\mu^2-r)}{\sigma^2}\rho}{(1-\rho^2)^{1/2}}$

and then

$$\omega^1_t = W^1_t + \frac{(\mu^1 - r)}{\sigma^1} t$$

So, under $\mathbb{Q}$ we have both of $W^i_t + \dfrac{(\mu^i - r)}{\sigma^i} t$ are martingales. There is more! We want to calculate $\langle \omega^1, \omega^2\rangle^{(+)}_t$. In fact we go up a level of generality and examine $\langle \tilde{U}, \tilde{V}\rangle^{(+)}$ where U, V are $L^2(\mathbb{P})$ martingales and $\tilde{U}$ and $\tilde{V}$ are defined as above. So

(+) Under $\mathbb{Q}$

$$\tilde{U}_t = U_t - \int_0^t X^1_s\, d\langle U, B^1\rangle_s - \int_0^t X^2_s\, d\langle U, B^2\rangle_s$$

and $\tilde{V}_t$ "the same". Using the integration by parts formula; under $\mathbb{P}$,

$$\tilde{U}_t\tilde{V}_t = U_0V_0 + \int_0^t U_s dV_s - \int_0^t U_s X_s^1 d\langle V, B^1\rangle_s - \int_0^t U_s X_s^2 d\langle V, B^2\rangle_s$$
$$+ \int_0^t V_s dU_s - \int_0^t V_s X_s^1 d\langle U, B^1\rangle_s - \int_0^t V_s X_s^2 d\langle U, B^2\rangle_s$$
$$+ \langle U, V\rangle_t .$$

So,

$$\tilde{U}_t\tilde{V}_t - \langle U, V\rangle_t = U_0V_0 + \int_0^t V_s dU_s + \int_0^t U_s dV_s - \int_0^t U_s X_s^1 d\langle V, B^1\rangle_s$$
$$- \int_0^t U_s X_s^2 d\langle V, B^2\rangle_s - \int_0^t V_s X_s^1 d\langle U, B^1\rangle_s$$
$$- \int_0^t V_s X_s^2 d\langle U, B^2\rangle_s$$

Now $(\tilde{U}\tilde{V} - \langle U,V\rangle)Z(X)$ is, using the integration by parts formula,

$$U_0V_0 + \int_0^t (\tilde{U}_s\tilde{V}_s - \langle U,V\rangle_s) dZ(X)_s + \int_0^t Z(X)_s V_s dU_s +$$
$$\int_0^t Z(X)_s U_s dV_s - \int_0^t Z(X)_s U_s X_s^1 d\langle V, B^1\rangle_s - \int_0^t Z(X)_s U_s X_s^2 d\langle V, B^2\rangle_s$$
$$- \int_0^t Z(X)_s V_s X_s^1 d\langle U, B^1\rangle_s - \int_0^t Z(X)_s V_s X_s^2 d\langle U, B^2\rangle_s +$$
$$\langle \tilde{U}\tilde{V} - \langle U,V\rangle, Z(X)\rangle_t .$$

The last term above, the cross-variation, is equal to

$$\left\langle \int_0^t V_s dU_s + \int_0^t U_s dV_s \,, \int_0^t X_s^1 Z(X)_s dB_s^1 + \int_0^t X_s^2 Z(X)_s dB_s^2 \right\rangle$$

which is,

$$\int_0^t V_s X_s^1 Z(X)_s d\langle U, B^1\rangle_s + \int_0^t V_s X_s^2 Z(X)_s d\langle U, B^2\rangle_s +$$

$$\int_0^t U_s X_s^1 Z(X)_s d\langle V, B^1\rangle_s + \int_0^t U_s X_s^2 Z(X)_s d\langle V, B^2\rangle_s$$

These terms cancel with those in the expression for $(\tilde{U}\tilde{V} - \langle U, V\rangle)Z(X)$ leaving

$$\int_0^t (U_s V_s - \langle U, V\rangle_s) dZ(X)_s + \int_0^t Z(X)_s V_s dU_s + \int_0^t Z(X)_s U_s dV_s$$

$+ U_0 V_0$. Which is a $\mathbb{P}$-martingale.(*) So $\langle U, V\rangle_{\mathbb{P}} = \langle \tilde{U}, \tilde{V}\rangle_{\mathbb{Q}}$ a.s. $\mathbb{P}$ and $\mathbb{Q}$, by the uniqueness of the cross-variation.

Accordingly,

$$\langle W^1, W^2\rangle_{\mathbb{Q}} = \left\langle W_t^1 + \left(\frac{\mu^1 - r}{\sigma^1}\right)t, W_t^2 + \left(\frac{\mu^2 - r}{\sigma^2}\right)t\right\rangle_{\mathbb{P}}$$

$$= \langle W_t^1, W_t^2\rangle_{\mathbb{P}}$$

$$= \rho t$$

So we see that under $\mathbb{Q}$ the discounted assets are $\mathbb{Q}$ martingales so that, under $\mathbb{Q}$, for $i = 1, 2$,

$$S_t^i = S_0^i + \int_0^t r S_s^i ds + \int_0^t \sigma^i S_s^i dW_s^i$$

* So $\tilde{U}\tilde{V} - \langle U, V\rangle$ is a $\mathbb{Q}$ martingale and $\langle U, V\rangle$ must be $\langle \tilde{U}, \tilde{V}\rangle$ under $\mathbb{Q}$.

and $\langle W^1, W^2\rangle_t = \rho t$. Moreover under $\mathbb{Q}$,

$$\langle W^i, W^i\rangle_t^{\mathbb{Q}} = \left\langle W^i + \left(\frac{\mu^i - r}{\sigma^i}\right)t, W^i + \left(\frac{\mu^i - r}{\sigma^i}\right)t\right\rangle_t^{\mathbb{P}}$$
$$= \langle W^i, W^i\rangle_t^{\mathbb{P}}$$
$$= t I_2$$

So, by Levy's Theorem, W^i is a $\mathbb{Q}$ Brownian Motion.

You may be asking, "why did you draw attention to the multi-dimesional versions of Levy's and Girsanov's results when you don't seem to have used them?". Well, for precisely that reason. It would be natural in a multi-asset model to attempt to employ multi-dimesional Levy, Girsanov, etc. But this can lead one astray. One problem that emerges is that we might apply Girsanov to a pair $(W^1+\lambda_1 t, W^2+\lambda_2 t)$ with $\langle W^1, W^2\rangle = \rho t$. But we would generated a 2-dimensional BM, with independent components, destroying the correlation between the original BM's.